

I. *Of the Bases of the Cells wherein the Bees deposite their Honey. Part of a Letter from Mr. Mac Laurin, Professor of Mathematics at Edinburgh, and F. R. S. to Martin Folkes, Esq; Pr. R. S.*

June 30. 1743.

Presented to  
the Royal So-  
ciety Nov. 3.  
1743.

THE Sagacity of the *Bees* in making their Cells of an hexagonal Form, has been admired of old; and that Figure has been taken notice of, as the best they could have pitched upon for their Purposes: But a yet more surprizing Instance of the Geometry of these little *Insects* is seen in the Form of the Bases of those Cells, discovered in the late accurate Observations of Monsieur *Maraldi* and Monsieur *de Reaumur*, who have found those Bases to be of that Pyramidal Figure, that requires the least Wax for containing the same Quantity of Honey, and which has at the same time a very remarkable Regularity and Beauty, connected of Necessity with its Frugality.

These Bases are formed from Three equal *Rhombus's*, the obtuse Angles of which are found to be the Doubles of an Angle that often offers itself to Mathematicians in Questions relating to *Maxima* and *Minima*; that is, the Angle, whose Tangent is to the *Radius*, as the Diagonal is to the Side of the Square. By this Construction, of the Six solid Angles at the Base that correspond to the Angles of the

E e e e

Hexa-

Hexagon, Three are equal as well to each other, as to the solid Angle at the *Apex* of the Figure, each of which solid Angles is respectively formed from Three equal plane obtuse Angles: And the other Three solid Angles are also equal to each other, but severally formed each from Four equal plane acute Angles, Supplements to the former obtuse ones.

By this Form the utmost Improvement is made of their Wax, of which they are on all Occasions very saving, the greatest Regularity is obtained in the Construction, and with a particular Facility in the Execution; as there is one sort of Angle only with its Supplement, that is required in the Structure of the whole Figure.

Monsieur *Maraldi* \* had found by Mensuration, that the obtuse Angles of the *Rhombus's* were of 110 Degrees nearly; upon which he observed, that if the Three obtuse Angles which formed the solid Angles above-mentioned, were supposed equal to each other, they must each be of  $109^{\circ}.28'$ ; from whence it has been inferred, that this last was really the true and just Measure of them: And lately Monsieur *de Reaumur* † has informed us, that Mr. *Koëning* having, at his Desire, sought what should be the Quantity to be given to this Angle, in order to employ the least Wax possible in a Cell of the same Capacity, that Gentleman had found, by a higher *Geometry* than was known to the Antients, by the Method of *Infinifimals*, that the Angle in question ought in this Case to be of  $109^{\circ}.26'$ . And we shall now make

---

\* *Memoires de l'Acad. Royale des Sciences*, 1712.

† *Memoires sur les Insectes*, Tom. V.

it appear from the Principles of common *Geometry*, that the most advantageous Angle for these *Rhombus's* is indeed, on that Account also, the same which results from the supposed Equality of the Three plane Angles that form the above-mentioned solid ones.

Let  $GN$  and  $NM$  represent any Two adjoining Sides of the Hexagon, that is, TAB. I.  
Fig 1. and 2. the Section of the Cell perpendicular to its Length. The Sides of the Cell are not complete Parallelograms as  $CGNK$ ,  $BMNK$ , but *Trapezia*  $CGNE$ ,  $BMNE$ , to which a *Rhombus*  $CEBe$ , is fitted at  $E$ , and that has the opposite Point  $e$  in the *Apex* of the Figure, so that Three *Rhombus's* of this kind, with Six *Trapezia*, may complete the Figure of the Cell. Let  $O$  be the Centre of the Hexagon, of which  $CK$  and  $KB$  are adjoining Sides; join  $CB$  and  $KO$ , intersecting it in  $A$ ; and, because  $COB$  is equal to  $CKB$ , and  $KE$  equal to  $Oe$ , the Solid  $EBCK$  is equal to the Solid  $eBCO$ ; from which it is obvious, that the Solid Content of the Cell will be the same, where-ever the Point  $E$  is taken in the Right Line  $KN$ , the Points  $C, K, B, G, N$ , and  $M$ , being given. We are therefore to inquire where the Point  $E$  is to be taken in  $KN$ , so that the *Area* of the *Rhombus*  $CEBe$ , together with that of the Two *Trapezia*  $CGNE$ ,  $ENMB$ , may form the Least Superficies. Because  $Ee$  is perpendicular to  $BC$  in  $A$ , the *Area* of the *Rhombus* is  $AE \times BC$ , that of the *Trapezia*  $CGNE$ ,  $ENMB$ , is  $\frac{CG + EN}{2} \times KC$ ; these, added to the *Rhombus*, amount to  $AE \times BC + 2KN \times KC - KE \times KC$ ; and because  $2KN \times KC$  is invariable, we are to

E e e e 2

in-

inquire, when  $AE \times BC - KE \times KC$  is a *Minimum*?

Suppose the Point  $L$  to be so taken upon  $KN$ , that  $KL$  may be to  $AL$  as  $KC$  is to  $BC$ . From the Centre  $A$  describe in the Plane  $AKE$  with the Radius  $AE$ , an Arc of a Circle  $ER$  meeting  $AL$ , produced, if necessary, in  $R$ ; let  $EV$  be perpendicular to  $AR$  in  $V$ , and  $KH$  be perpendicular to the same in  $H$ ; then the Triangles  $LEV, LKH, LAK$ , being similar, we have  $LV:LE::LH:LK::LK:LA::$  (by the Supposition last made)  $KC:BC$ . Hence, when  $E$  is between  $L$  and  $N$ , we have  $LH+LV(=VH):LK+LE(=KE)::KC:BC$ ; and when  $E$  is between  $K$  and  $L$ , we have  $LH-LV(=VH):LK-LE(=KE)::KC:BC$ ; that is, in both Cases we have  $KE \times KC = VH \times BC$ ; and consequently  $AE \times BC - KE \times KC = AE \times BC - VH \times BC = \overline{AE - VH} \times BC = \overline{AR - VH} \times BC = \overline{AH + VR} \times BC$ ; which, because  $AH$  and  $BC$  do not vary, is evidently Least when  $VR$  vanishes, that is, when  $E$  is upon  $L$ . Therefore  $CLBL$  is the *Rhombus* of the most advantageous Form in respect of Frugality, when  $KL$  is to  $AL$  as  $KC$  is to  $BC$ . This is the same Method by which we have elsewhere determined the *Maxima* and *Minima*, in the Resolution of several Problems that have usually been treated in a more abstruse Manner. See *Treatise of Fluxions*, Art. 572, &c.

Now because  $OK$  is bisected in  $A$ ,  $KC^2 = OK^2 = 4AK^2$ ; and  $AC^2 = 3AK^2$ , or  $BC = 2AC = 2\sqrt{3} \times AK$ ; consequently  $KC:BC::2AK:2\sqrt{3} \times AK::1:\sqrt{3}$ , and  $KL:AL::(KC:BC)::1:\sqrt{3}$ ,

1 :  $\sqrt{3}$ , or  $AL : AK :: \sqrt{3} : \sqrt{2}$ ; and (because  $AK : AC :: 1 : \sqrt{3}$ )  $AL : AC :: 1 : \sqrt{2}$ ; that is, the Angle  $CLA$  is that, whose Tangent is to the *Radius* as  $\sqrt{2}$  is to 1, or as 14142135 to 10000000; and therefore is of  $54^{\circ}. 44'. 08''$ , and consequently the Angle of the *Rhombus* of the Best Form is that of  $109^{\circ}. 28'. 16''$ .

By this Solution it is further easy to estimate what their Savings may amount to upon this Article, in consequence of this Construction. Had they made the Base flat, and not of the pyramidal Form described above, then, besides completing the Parallelograms  $CGNK$  and  $BMNK$ , the Surface of the Base had been  $3 CB \times AK$ ; what they really do form amounts in Surface to the same Parallelograms, and  $3 CB \times AH$ : the Savings therefore amount to  $3 CB \times AK - AH$   
 $= 3 CB \times AH \times \frac{\sqrt{1} - \sqrt{2}}{\sqrt{2}}$ , which is almost a Fourth-part of the Pains and Expence of Wax, they bestow above what was necessary for completing the parallelogram Sides of the Cells: And at the same time they seem also to have other Advantages from this Form, besides the saving of their Wax; such as a greater Strength of the Work, and more Convenience for moving in these larger solid Angles.

It remains that we should shew, that the plane Angles  $CLB$ ,  $CLN$ , and  $BLN$ , are equal to each other. We before found, that  $KL : AL :: KC : BC :: KA : (\frac{1}{2} KC) AC$ ; consequently  $KL : KA :: AL : AC$ , and the Triangles  $LKA$ ,  $LAC$ , are similar: Therefore  $LK : AL :: AL : LC :: KC : BC :: 1 : \sqrt{3}$ , and  $LC = 3 LK$ . With the Centre  $L$  and *Radius*  $LC$ , describe in the Plane  $CGNK$  the Semicircle  $DGP$ , meeting the Line  $KN$ , in  $D$  and  $P$ ;

TAB. I.  
Fig. 3.

join

join  $CP$  and  $CD$ , and let  $LQ$  be perpendicular to  $CP$  in  $Q$ , then will the Angle  $CDK$  be equal to  $QLP$ , and we shall have  $PQ : LQ :: PC : DC :: \sqrt{PK} : \sqrt{DK} :: \sqrt{LC+LK} : \sqrt{LC-LK} :: \sqrt{4} : \sqrt{2} :: \sqrt{2} : 1 :: AC : AL$ . Consequently the Angle  $QLP = ALC$ , and  $CLP = CLB$ , or the obtuse Angle of the *Rhombus*  $CLBl$  is equal to  $CLP$ , the obtuse Angle of the *Trapezium*; and consequently, the Three plane Angles that form the solid Angle at  $L$ , or the *Apex* at  $l$ , are equal to each other: From which it is obvious, that the Four acute plane Angles, which form the solid Angle at  $C$  or  $B$ , are likewise equal among themselves.

Though Monsieur *Maraldi* had found, by his Mensuration, these obtuse Angles to be of about 110 Degrees; the small Difference between this and the  $109^{\circ}. 28'. 16''$ , just found by Calculation, seems to have been either accidental, or owing to the Difficulty of measuring such Angles with Exactness: Besides that he seems to admit the real Equality of the several plane Angles, that form as well the *Apex*, as the other solid ones we have been treating of. And, as to the small Difference between our Angle and that determined by Mr. *Koëning*, who first considered this Problem, but has not yet published his Demonstration of it, that can only be owing to his not carrying on his Computation so far, and would scarcely have been worth the mentioning, were it not yet in Favour of the Practice of these industrious little Insects; and did it not therefore give us ground to conclude, that in general, and when the particular Form and Circumstance of the Honey-comb does not require a Variation from their Rule,

the

the *Bees* do truly construct their Cells of the best Figure, and that not only nearly, but with Exactness; and that their Proceeding could not have been more perfect from the greatest Knowledge in *Geometry*. How they arrive at this, and how the wonderful Instinct in Animals is to be accounted for, is a Question of an higher Nature; but this is surely a remarkable Example of this Instinct, as it has suggested a Problem that had been overlooked by Mathematicians, though they have treated largely on the *Maxima* and *Minima*; and such an one, as has been thought to exceed the Compass of the common *Geometry*.

It may be worth while to add here, that if the Cells had been of any other Form than hexagonal, and the Bases had still been pyramidal, these must have been terminated by *Trapezia*, and not by *Rhombus's*, and therefore had been less regular, because *OA* and *AK* would have been unequal: Nor could there have been room for such an advantageous or frugal a Construction as that we have described, because the solid Content of the Cell would have increased with the Right Line *KE*. The Cells, by being hexagonal, are the most capacious, in proportion to their Surface, of any regular Figures that leave no Interstices between them, and at the same time admit of the most perfect Bases. Thus, by following what is best in one respect, unforeseen Advantages are often obtained; and what is most beautiful and regular, is also found to be most useful and excellent.

Fig. 1.

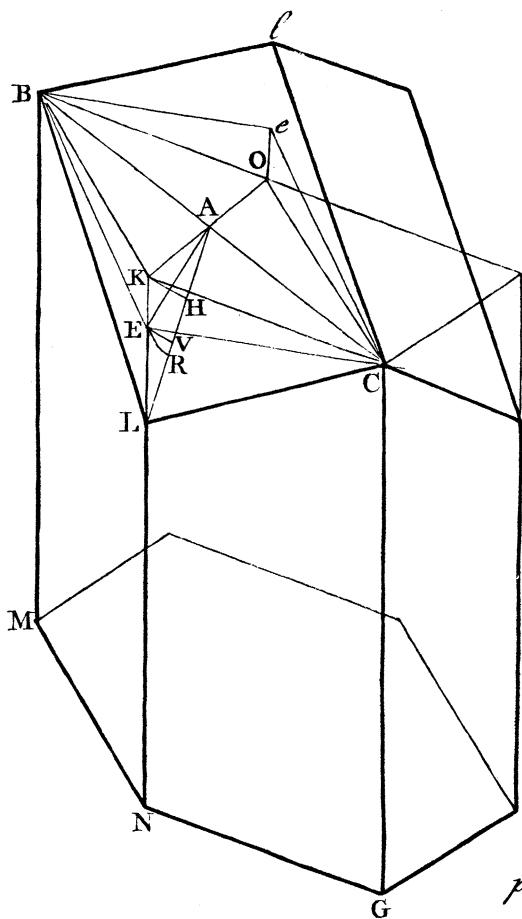
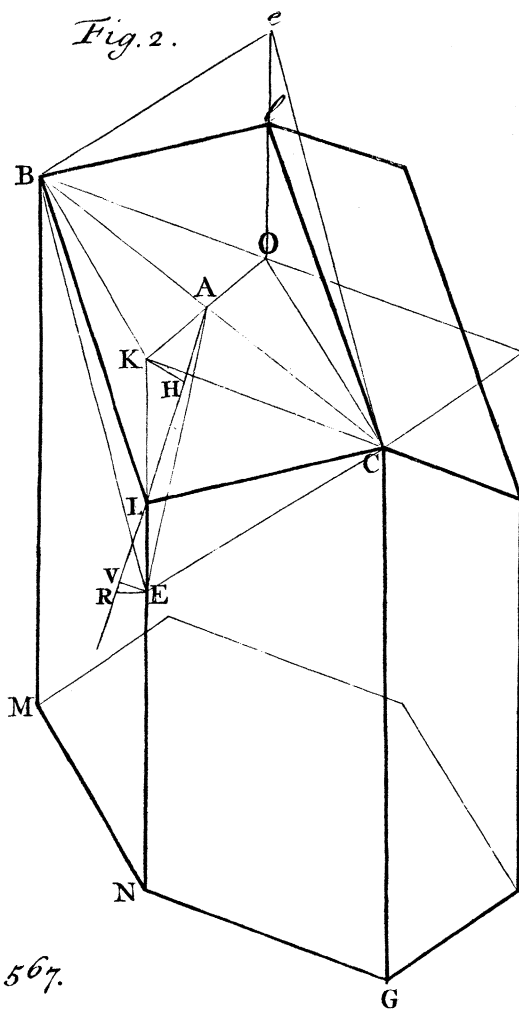
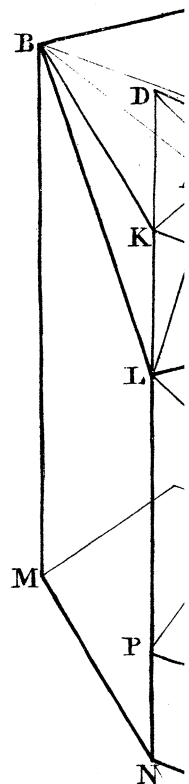


Fig. 2.



p. 567.



*Fig. 3.*

